Evolution of Dynamical Coupling in Scalar Tensor Theory from Noether Symmetry

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We present the gravitational coupling function $\omega(\phi)$ in the vacuum scalar-tensor theory as allowed by the Noether symmetry. We also obtain some exact cosmological solutions in the spatially homogeneous and isotropic background thereby showing that the attractor mechanism is not effective enough to reduce the theory to Einstein theory. It is observed that, asymptotically, the scalar tensor theory goes over to Einstein theory with finite value of ω. This work thus supports earlier works in this direction.

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I. INTRODUCTION

The scalar-tensor theory of gravity is the simplest generalization of general theory of gravity in which the gravitational interaction is mediated by long range scalar field ϕ in addition to the usual tensor field $g_{\mu\nu}$ present in Einstein's theory. The strength of the coupling between the scalar field and gravity, in general, is determined by the coupling function $\omega(\phi)$. Brans and Dicke [1] explored the simplest possible form of the scalar-tensor theory in which the coupling ω is a constant parameter and the value of ω is constrained by the classical test of general relativity [2]. In particular , the bending of light by the Sun and time delay experiments require $\omega > 500$ and the bounds on the anisotropy of the microwave background radiation gives $\omega \leq 30$. Thus it is natural to assume ω as a function of ϕ in the scalar-tensor theory to represent a viable model of the universe. Bergmann,Nordtvedt and Wagoner [3–5] generalized the scalar-tensor theory in which the scalar field has a dynamical coupling with gravity and /or an arbitrary self interaction. Further, the recent unification schemes [6] of fundamental interaction based on supergravity or superstrings naturally associate a long range scalar partners to the usual tensor field present in Einstein's gravity. In the weak energy limit different unification schemes reduce gravity theories having a non minimal coupling $\omega(\phi)$ between a scalar field ϕ with curvature R of the geometry.

It is generally believed that the Brans Dicke theory goes over to general relativity in the $\omega \to \infty$ limit. But recently Romero and Barros [7], Banerjee and Sen [8] have shown that the Brans Dicke theory does not go over to general relativity in the large ω limit if the trace of the energy momentum tensor describing all fields other than Brans Dicke scalar field ϕ is zero. Recently Santiago et al [9] presented some models on the scalar-tensor cosmology in FRW spacetime background. In one case they show that the attractor mechanism is not effective enough to reduce the theory (asymptotically as $t \to \infty$) towards a final state indistinguishable from general relativity for massless scalar field. Santiago et al [9] used the conformal transformation and assumed an arbitrary special form of dynamical coupling function $\omega(\phi)$ to justify the above attractor mechanism. So far there is no unique way to find the functional form of $\omega(\phi)$ in the general scalar-tensor gravity theories. However Noether symmetry approach [10] , [11], [12], [13] to the scalar-tensor theory allows one to obtain the form of $\omega(\phi)$ from the symmetry arguments. We propose in this work a way to calculate $\omega(\phi)$ from the Noether symmetry of the Lagrangian of the theory and thereby asking what happens dynamically to the asymptotic limit. This will ensure one way the consistency of the Noether symmetry approach and to verify the other recent results [7–9] in this direction.

We present the dynamical form of $\omega(\phi)$ in a scalar-tensor theory without any self interaction of the scalar field ϕ and devoid of other matter fields except the scalar field ϕ. We consider the action

$$A = \int \quad d^4x\sqrt{-g}[\phi R - \omega(\phi)\frac{\phi_{,\mu}\phi^{,\mu}}{\phi}], \tag{1}$$

where R is the Ricci scalar.

The principle used by de Ritis et al [10], [11] in calculating the unknown functions (e.g., $\omega(\phi)$) in the Lagrangian is that the action is invariant under transformation corresponding to the Noether symmetries in the spatially homogeneous and isotropic background. Using this technique, we determine $\omega(\phi)$ and then present a few exact solutions in the spatially homogeneous and isotropic FRW spacetime. In one class of the solutions, the theory does not go over to the Einstein equation with $\omega \to \infty$ asymptotically as $t \to \infty$, rather we obtain Einstein equation for finite value of ω for open three space section as $t \to \infty$. In a special case we obtain a solution which reduces to the Einstein equation with $\omega \to \infty$ asymptotically. This work thus also supports the claim made in ref. [7], [8]. Our work has the advantage that we determine $\omega(\phi)$ dynamically rather than setting it to a constant (as in ref.7,8) or having an ad hoc choice [9] for $\omega(\phi)$.

II. FORM OF $\omega(\phi)$ FROM NOETHER SYMMETRIES

In spatially homogeneous and isotropic background the point Lagrangian from the action (1) is

$$L = -6\phi a\dot{a}^2 - 6a^2\dot{a}\dot{\phi} + \omega(\phi)\frac{a^3\dot{\phi}^2}{\phi} + 6ka\phi, \tag{2}$$

where $k = 0, \pm 1$ and an overdot denotes derivative with respect to proper time t and $a(t)$ is the scale factor. Now in a given dynamical system, if the Lagrangian is independent of one of the configuration space variable, then its canonical momenta is a constant of motion and we have a Noether symmetry corresponding to above cyclic co-ordinate. In the absence of such trivial symmetry we can use the Noether symmetry approach as followed by de Ritis et al [10], and Capozziello et al [11] to determine the vector field $\mathbf{X}$, i.e., a Noether symmetry (if they exist) for the dynamics derived by the point Lagrangian L. In the Lagrangian (2) we consider the configuration space $Q \equiv (a, \phi)$, whose tangent space $TQ \equiv (a, \dot{a}, \phi, \dot{\phi})$. The infinitesimal generator of the Noether symmetry, i.e., the lift vector $\mathbf{X}$ is now written as

$$\mathbf{X} = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial \phi} + \frac{d\alpha}{dt}\frac{\partial}{\partial \dot{a}} + \frac{d\beta}{dt}\frac{\partial}{\partial \dot{\phi}} \tag{3}$$

where α, β are functions of a and ϕ. The existence of the Noether symmetry implies the existence of a vector field $\mathbf{X}$, such that

$$\mathcal{L}_{\mathbf{X}}L = 0, \tag{4}$$

where $\mathcal{L}_{\mathbf{X}}$ stands for Lie derivative with respect to $\mathbf{X}$. Now, from (4) we can determine the vector field $\mathbf{X}$ and as a consequence $\omega(\phi)$ can be found out. Equation (4) gives an expression of second degree in $\dot{a}$ and $\dot{\phi}$, and whose coefficients are zero to satisfy (4). Thus from (4) we have

$$k(\phi\alpha + a\beta) = 0, \tag{5}$$

$$\phi\alpha + a\beta + 2a\phi\frac{\partial \alpha}{\partial a} + a^2\frac{\partial \beta}{\partial a} = 0, \tag{6}$$

$$3\omega\alpha - \frac{\omega a\beta}{\phi} - 6\phi\frac{\partial \alpha}{\partial \phi} + 2\omega a\frac{\partial \beta}{\partial \phi} + a\beta\frac{\partial \omega}{\partial \phi} = 0, \tag{7}$$

$$6\alpha + 3a\frac{\partial \alpha}{\partial a} + 6\phi\frac{\partial \alpha}{\partial \phi} + 3a\frac{\partial \beta}{\partial \phi} - \frac{\omega a^2}{\phi}\frac{\partial \beta}{\partial a} = 0. \tag{8}$$

Now according to the values of k we have two distinct cases. So, first we consider non-vanishing three space curvature, thus having $k \neq 0$ and we have from (5) and (6)

$$\alpha = -\frac{a\beta}{\phi}, \tag{9}$$

and

$$\beta = n(\phi)a^2, \tag{10}$$

where $n(\phi)$ is function of ϕ only.Using (9) and (10) in (8) we get

$$\frac{dn}{d\phi} = \frac{n(2\omega + 3)}{3\phi}. \tag{11}$$

Now using (9),(10) and(11) in (7) we get

$$\omega(\phi) = \frac{3}{2}{\phi_0}^2(\phi^2 - {\phi_0}^2)^{-1} \tag{12}$$

where ϕ_0 is a constant. From (11), (12) ,(9) and (10) we get

$$\alpha = -(\phi^2 - {\phi_0}^2)^{\frac{1}{2}}(a\phi)^{-1}, \tag{13}$$

$$\beta = (\phi^2 - {\phi_0}^2)^{\frac{1}{2}}a^{-2}. \tag{14}$$

The vector field $\mathbf{X}$ is thus determined by (13),(14). The existence of the symmetry $\mathbf{X}$ gives us a constant of motion, via the Noether theorem. The constant of motion is given by

$$\Sigma = i_{\mathbf{X}}\theta_L = (\phi^2 - {\phi_0}^2)^{\frac{1}{2}}[6\dot{a} + 2a(\omega + 3)\frac{\dot{\phi}}{\phi}], \tag{15}$$

where the Cartan one form is given by

$$\theta_L = \frac{\partial L}{\partial \dot{a}}da + \frac{\partial L}{\partial \dot{\phi}}d\phi \tag{16}$$

Thus we see that the existence of the Noether symmetry allows us to determine the dynamical coupling function $\omega(\phi)$, as given in (12).

Now we consider the vanishing three space curvature case. For $k = 0$,the equation (9) (valid when $k \neq 0$) may not be valid in general. So we consider the solutions of α, β and ω from (6),(7) and (8) only. The solutions are

$$\alpha = \alpha_0 a^{\epsilon - \frac{1}{2}}\phi^c(\phi^{\frac{s}{2}} + \phi^{\frac{-s}{2}})^d, \tag{17}$$

$$\beta = -\beta_0 a^{\epsilon - \frac{3}{2}}\phi^{c+1}(\phi^{\frac{s}{2}} + \phi^{\frac{-s}{2}})^d, \tag{18}$$

and

$$\omega = \frac{3\lambda}{2}[\frac{(\phi_0\phi)^s - 1}{(\phi_0\phi)^s + 1}] - (\epsilon^2 - \frac{1}{8}), \tag{19}$$

where

$$\alpha_0 = K(\epsilon - \frac{1}{2})^{-1},$$
$$\beta_0 = K(\epsilon - \frac{1}{2})^{-2},$$
$$c = (\frac{3}{2} - \epsilon)(\epsilon^3 - \frac{9\epsilon}{8} - \frac{1}{2}),$$
$$d = (\epsilon^2 - \frac{3\epsilon}{2})(2\epsilon^2 - 3)^{-1},$$
$$s = 2\lambda(3 - 2\epsilon^2),$$
$$\lambda = \epsilon^{-\frac{1}{2}}(\frac{1}{4} - \epsilon^2)^{\frac{1}{2}}(\epsilon^2 - \frac{1}{8})^{\frac{1}{2}}, \tag{20}$$

where k, ϕ_0 and ϵ are constant of integration. The solutions (17), (18), (19) represent a physically acceptable Noether symmetry provided $\frac{1}{8} < \epsilon^2 < \frac{1}{4}$ for $k = 0$ (that follows from the last equation of (20)). The constant of motion corresponding to the symmetry is evaluated and is given by

$$\Sigma = (\epsilon - \frac{1}{2})\epsilon^{-1}\beta(6a^2\dot{a} - 3a^3\frac{\dot{\phi}}{\phi}) + \beta(-6a^2\dot{a} + 2\omega a^3\frac{\dot{\phi}}{\phi}). \tag{21}$$

III. THE FIELD EQUATION AND SOLUTION

We consider the solution of the field equations for non-vanishing three space curvature only, as the solution in closed form can be obtained in such cases easily. The field equations from (2) are

$$\frac{\ddot{a}}{a} - \frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi} + \frac{\omega}{3}\frac{\dot{\phi}^2}{\phi^2} = \frac{\dot{\phi}^2}{2(2\omega+3)\phi}\frac{d\omega}{d\phi}, \tag{22}$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = -\frac{d\omega}{d\phi}\frac{\dot{\phi}^2}{(2\omega+3)}, \tag{23}$$

and the constraint equation is

$$\dot{a}^2 + k = \frac{\omega a^2\dot{\phi}^2}{6\phi^2} - a\dot{a}\frac{\dot{\phi}}{\phi}, \tag{24}$$

where ω is given by (12).The solutions of (23)-(24) are not easy to evaluate in the present form. So we introduce a new set of configuration space variable Q_1, Q_2 instead of old variables a and ϕ.The new variables are given by

$$Q_1 = \frac{\phi}{\phi_0^2}a^2(\phi^2 - \phi_0^2)^{\frac{1}{2}}, \tag{25}$$

$$Q_2 = (a\phi)^{r_1}, \tag{26}$$

where r_1 is an arbitrary constant. In the new set of configuration space variables, the Lagrangian (2) transforms to

$$L = \frac{3}{2}\phi_0^2\dot{Q_1}^2 Q_2^{-\frac{1}{r_1}} - 6(\phi_0 r_1)^{-2}Q_2^{\frac{3}{r_1}-2}\dot{Q_2}^2 + 6kQ_2^{\frac{1}{r_1}}. \tag{27}$$

Here Q_1 appears as a cyclic variable and hence implies the existence of the symmetry. The dynamical equations written in terms of Q_1 and Q_2 are

$$2Q_2\ddot{Q}_2 + (\frac{3}{r_1} - 2)\dot{Q_2}^2 = \frac{r_1}{4}\phi_0^4\dot{Q_1}^2 Q_2^{2-\frac{4}{r_1}} - kr_1\phi_0^2 Q_2^{2-\frac{2}{r_1}}, \tag{28}$$

$$\Sigma = 3\phi_0^2 \frac{\dot{Q}_1}{Q_2^{\frac{1}{r_1}}}, \tag{29}$$

where Σ is the constant of motion of eqn.(15). The constraint equation is

$$\phi_0^2\dot{Q_1}^2 Q_2^{-\frac{1}{r_1}} - \frac{4}{r_1^2\phi_0^2} Q_2^{\frac{3}{r_1}-2}\dot{Q_2}^2 - 4kQ_2^{\frac{1}{r_1}} = 0. \tag{30}$$

It is to be noted that the equation (15) can be transformed to (29) by using transformation (25) and (26). The solution of above field equations are

$$Q_1 = \frac{\Sigma}{3\phi_0^2}(\frac{1}{2}Q_0 t^2 + c_0 t + c_1), \tag{31}$$

and

$$Q_2 = (Q_0 t + c_0)^{r_1}, \tag{32}$$

where c_0 and c_1 are integration constants and

$$Q_0^2 = \frac{\Sigma^2}{36} - k\phi_0^2. \tag{33}$$

These are the vacuum solutions of the scalar-tensor theories in spatially homogeneous and isotropic background for nonvanishing three space curvature $k = \pm 1$.Now using (31) and (32) in (25), (26) and (12), the scale factor , scalar field and the coupling function are given by

$$a^2(t) = (Q_0 t + c_0)^2\phi_0^{-2} - \Sigma^2(3\phi_0)^{-2}(Q_0\frac{t^2}{2} + c_0 t + c_1)^2(Q_0 t + c_0)^{-2}, \tag{34}$$

$$\phi^2(t) = \phi_0^2[1 - \frac{\Sigma^2}{9(Q_0 t + c_0)^4}(Q_0 t^2/2 + c_0 t + c_1)^2], \tag{35}$$

$$2\omega + 3 = 27\Sigma^{-2}(Q_0 t + c_0)^4(Q_0 t^2/2 + c_0 t + c_1)^{-2}. \tag{36}$$

The solutions (35)-(36) represent a class of solutions of the scalar-tensor theories depending on the value of the integration constants namely c_0, c_1 and Σ where the integration constant Q_0 is determined by (33). The set of solutions (33)-(36) are valid for non-vanishing Σ. Now we shall consider the behaviour of above solutions for different values of c_0 and c_1.
Case I: $c_1 = 0 = c_0$ and $\Sigma \neq 0$.
The solution is

$$a^2(t) = -kt^2, \phi^2 = -Q_0^2/k \quad and \quad \omega = -54k\phi_0^2/\Sigma^2. \tag{37}$$

We get the trivial solution of vacuum Einstein equation, though we do not have $\omega \to \infty$.
Case II: When $c_1 = 0$, but $c_0 \neq 0, \Sigma \neq 0$, the equations (34)-(36) represent an expanding universe. At the epoch $t = 0$, the scale factor and the scalar field ϕ are finite (positive non-zero), but the coupling function ω diverges. Asymptotically as $t \to \infty$, we have

$$\begin{aligned} a^2(t \to \infty) &\approx -kt^2, \\ \phi^2(t \to \infty) &\approx \phi^2(t = 0) + \Sigma^2/36, \\ \omega(t \to \infty) &\approx -54k\phi_0^2/\Sigma^2. \end{aligned} \tag{38}$$

Thus for physically realizable solution the three space section has to be open. It is important to note that the value of the scalar field ϕ is increasing with expansion of the universe, i.e., the value of the effective Newtonian gravitational constant G_N is decreasing with the expansion of the universe. Further, we note that the solution asymptotically goes over to the vacuum Einstein equation for finite value of the coupling function ω, so the final state of the universe is distinguishable from the Einstein gravity.

Case III: $c_1 \neq 0, c_0 \neq 0$, and $\Sigma \neq 0$.

The solution is acceptable for $t > 0$ provided the integration constants satisfy $(c_0^2 - \frac{\Sigma^2 c_1^2}{9c_0^2}) > 0$ and the three space section is open i.e., $k = -1$. In this case a^2, ϕ^2, ω are well behaved for $t \geq 0$ and unphysical at $t = -c_0/Q_0$. This case also represents an expanding universe in the region $t \geq 0$. Asymptotically as $t \to \infty$ we find

$$\begin{aligned} a^2(t \to \infty) &\approx -kt^2, \\ \phi^2(\to \infty) &= -Q_0^2/k = \phi_0^2 - \frac{\Sigma^2}{36k}, \\ \omega(t \to \infty) &= -54k\frac{\phi_0^2}{\Sigma^2}. \end{aligned} \tag{39}$$

So this case goes over to the vacuum Einstein equation for finite value of the coupling function ω, which is in direct contrast with the state of the universe at $t \to \infty$(characterized by $\omega \to \infty$) in the scalar-tensor theory for non-vanishing trace of the energy momentum tensor.

Case IV: $\Sigma = 0$.

Now we consider a special case in which the constant of motion $\Sigma = 0$ in equation (15) and solve the dynamical equations (22-24) without introducing new set of configuration space variables. Using (15) in equation (22) with $\Sigma = 0$ we have

$$\frac{\ddot{\phi}}{\dot{\phi}} - \frac{\dot{\phi}}{2\phi} - \frac{5\phi\dot{\phi}}{2(\phi^2 - \phi_0^2)} = 0, \tag{40}$$

whose first integral is

$$\dot{\phi} = \lambda\sqrt{\phi}(\phi^2 - \phi_0^2)^{5/4}, \tag{41}$$

where λ is an integration constant. The solution of (41) is

$$\phi^2 = \phi_0^2\frac{\tau^4}{\tau^4 - 1}, \tag{42}$$

where $\tau = \tau_0 - \lambda\phi_0^2 t/2, \tau_0$ being an integration constant. The scale factor is

$$a^2 = a_0^2\frac{\tau^4 - 1}{\tau^2}. \tag{43}$$

The constants ϕ_0, a_0, λ are not independent, rather related by, using (42),(43) in (23),

$$a_0^2 = -4k(\phi_0^2\lambda)^{-2}. \tag{44}$$

Thus for a physically realizable solution, the three space section must be open (i.e.,$k = -1$). The coupling function ω then becomes

$$\omega = \frac{3}{2}(\tau^4 - 1). \tag{45}$$

It is observed that the solution (43) is acceptable for $|\tau| > 1$ and the state of the universe asymptotically (as $|\tau| \to \infty$) is indistinguishable from the vacuum Einstein equation characterized by $\phi = \phi_0$ and $\omega \to \infty$. So $\phi = \phi_0$ is an attractor of the equation of motion.

IV. DISCUSSION

The main object of this paper is to investigate whether the vacuum scalar-tensor theory goes over to GTR or not at the asymptotic $t \to \infty$ limit. Our work supports the recent claim of Romero and Barros [7], Banerjee and Sen [8] and Santiago et al [9] that the final state of the universe (at $t \to \infty$) in the scalar-tensor theory is distinguishable from the Einstein's theory. This claim is due the fact that the scalar-tensor theory does not always go over to the Einstein's theory in the $\omega \to \infty$ limit, rather GTR may be recovered from the scalar-tensor theory at a finite value of ω.

Romero and Barros [7] worked out some examples, later Banerjee and Sen [8] argued from an order of estimate calculation to obtain the GTR from the Brans-Dicke theory at $\omega \to$ large value if the trace of the energy momentum tensor describing all fields other than Brans-Dicke field is zero. Naturally it is necessary to investigate the validity of the above results for a dynamical coupling of the scalar field and gravity in the scalar-tensor theory. Santiago et al [9] have shown that the attractor mechanism is ineffective for a massless scalar field by considering a conformal transformation along with an arbitrary choice of the dynamical coupling function $\omega(\phi)$.

In this work we consider the scalar-tensor theory without any self-interaction of the scalar field, i.e., our action is devoid of any matter field except the scalar field ϕ. This assumption is for mathematical simplicity. We determine the coupling function $\omega(\phi)$ from the Noether symmetry as allowed by the action (1). This is the advantage of our method over the ad hoc choice of $\omega(\phi)$ in ref.6. In our work the exact solution of the field equations singles out a dynamical constant of motion Σ, depending on whose value we have two distinct final states of the universe at $t \to \infty$. From the Noether symmetry we can identify Σ as the canonical conjugate momentum corresponding to coordinate Q_1. The value of the constant of motion Σ has to be determined from the boundary condition of the universe. Now, if we choose it as zero i.e., $\Sigma = 0$, then the scalar-tensor theory (case IV) goes over to the Einstein's theory with $\omega \to \infty$ limit asymptotically at $t \to \infty$.

However, if the constant of motion is non-zero, i.e., $\Sigma \neq 0$ (case I,II,III), we recover the Einstein's equation from the scalar-tensor theory with finite limiting value of ω at asymptotic limit $t \to \infty$ for the open universe. We are presently investigating the general validity of this statement considering the inclusion of the matter field in the scalar-tensor theory if $\Sigma = 0$ and $\Sigma \neq 0$ sectors do really matter in ascertaining the indistinguishable and distinguishable solutions of GTR at the asymptotic limit $t \to \infty$. This will help us understand the behaviour of the solutions at finite or at non-finite values of ω from symmetry arguments.

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